Generalized Geometry

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1 Abstract

In this work, we study generalized geometric structures such as Dirac structures and generalized complex structures introduced by Nigel Hitchin [8] and further developed by Gualtieri [9] . We introduce the Courant bracket which furnishes the integrability condition for a generalized complex structure on a manifold. We will introduce extended actions on courant algebroids and finally, we will define Moment maps for extended actions.

2 Introduction

This work was based on [1], we consider generalized geometric structures such as Dirac structures and generalized complex structures (see [2]). These are geometrical structures defined not on the tangent bundle of a manifold but on the sum $TM \oplus T^*M$ of the tangent and cotangent bundles (or, more generally, on an exact Courant algebroid).Such structures interpolate between many of the classical geometries such as symplectic, Poisson geometry, the geometry of foliations, and complex geometry.

In section 3 we introduce an extended notion of group action on a manifold preserving twisted Courant brackets. We start by recalling the definition and basic properties of Courant algebroids. The enhanced symmetry group of a Courant algebroid leads us to define extended actions and a generalized notion of moment map (visit [3] , [4]).

In subsection 4.1 we introduce the notion of a Courant algebra, and explain how it acts on a Courant algebroid in a way which extends the usual action of a Lie algebra by tangent vector fields.

This is important, for in the presence of a symmetry, a given geometrical structure may, under suitable conditions, pass to the quotient. Often, however, the quotient does not inherit the same type of geometry as the original space. For example, the quotient of a symplectic manifold by a symplectic S^1 action is never symplectic; rather it is endowed with a natural Poisson structure, whose leaves are the symplectic reduced spaces. Then it seeks to consider the reduction of generalized geometrical structures such as Dirac structures and generalized complex structures (see [5]). For many years physicists and mathematicians have studied the mysterious links between complex and symplectic geometry predicted by mirror symmetry. Therefore, the explicit unification of these two structures would seem an important step in understanding how they are connected. We will begin by giving a notion that it is a Lie algebroid to later introduce the Courant algebroid.

3 Lie algebroids

In this short section we introduce the notion of Lie algebroid, focusing our attention in providing several examples. For specific details the reader is recommended to visit [6].

Definition 1. *A* ***Lie algebroid*** *over a manifold M consists of a vector bundle A together with a bundle map*

$$\rho_A : A \to TM$$

and a Lie bracket $[\cdot,\cdot]_A$ *on the space of sections* $\Gamma(A)$*, satisfying the Leibniz identity*

$$[\alpha, f\beta]_A = f[\alpha,\beta]_A + \mathcal{L}_{\rho A(\alpha)}(f)\beta.$$

Let A be a Lie algebroid over M. We start by looking at the kernel and at the image of the anchor $\rho : A \to TM$. let $x \in M$ fix, the kernel is a Lie algebra: if $\alpha, \beta \in \Gamma(A)$ lie in $\mathrm{Ker}(\rho_x)$ when evaluated at x, the Leibniz identity implies that $[\alpha, f\beta](x) = f(x)[\alpha,\beta](x)$. Hence, there is a well defined bracket on $\mathrm{Ker}(\rho_x)$ such that

$$[\alpha,\beta](x) = [\alpha(x), \beta(x)]$$

Definition 2. *At any point* $x \in M$ *the Lie algebra* $\mathfrak{g}_x(A) = \ker(\rho_x)$ *is called the isotropy Lie algebra at* x*.*

Definition 3. *When* ρ *is surjective we say that* A *is a transitive Lie algebroid.*

We present an example where the Lie algebroid is defined by the Lie groupoid. In general, each Lie algebroid does not necessarily come from a Lie groupoid.

Example 1. *The Lie algebroid of a Lie grupoide* G *is the vector bundle* $A = \ ker(ds)$ *with* s *the source map of groupoid, together with the anchor*

$$\rho_A : A \to TM$$

obtained by restricting $d\boldsymbol{t} : T\mathcal{G} \to TM$ *to* $A \subset T\mathcal{G}$ *, and the Lie bracket* $[\cdot,\cdot]_A$ *on* $\Gamma(A)$*. Now we are going to describe how this Lie bracket* $[\cdot,\cdot]_A$ *is being defined on* $\Gamma(A)$*,*

- *The set of right-invariant vector fields in the groupoid, can be described as,*

  $$\mathfrak{X}^s_{inv}(\mathcal{G}) = \{X \in \Gamma(T^s\mathcal{G}) : X_{hg} = R_g(X_h), \forall (h,g) \in \mathcal{G}_2\}$$

  denoting $R_g : s^{-1}(\boldsymbol{t}(g)) \to s^{-1}(s(g))$ *to the differential* R_g *of at point* h *of the fiber* $s^{-1}(\boldsymbol{t}(g))$*. Note that, given* $a \in \Gamma(A)$*, the formula* $\tilde{\alpha}_g = R_g(\alpha_{\boldsymbol{t}(g)})$ *clearly defines a right invariant vector field. Conversely, any vector field* $X \in \mathfrak{X}^s_{inv}(\mathcal{G})$ *arises in this way: the invariance of* X *shows that* X *is determined by its values at the points in M:*

  $$X_g = R_g(X_y) \quad \forall g : x \to y$$

  i.e., $X = \tilde{\alpha}$ *where* $\alpha := X|_M \in \Gamma(A)$*. The Lie bracket on* A *is the* ***Lie bracket on*** $\Gamma(A)$ *is obtained from the Lie bracket on the set of right-invariant vector fields in the groupoid* $\mathfrak{X}_{inv}(\mathcal{G})$ *under the isomorphism* $\Gamma(A) \xrightarrow{\sim} \mathfrak{X}^s_{inv}(\mathcal{G})$ *with* $\alpha \to \tilde{\alpha}$ *. Hence this new bracket on* $\Gamma(A)$ *which we denote by* $[,]_A$ *(or simply* $[,]$*, when there is no risk of confusion) is uniquely determined by the formula:*

  $$\widetilde{[\alpha,\beta]}_A = [\tilde{\alpha}, \tilde{\beta}],$$

  where for all $\alpha, \beta \in \Gamma(A)$ *and all* $f \in C^\infty(M)$*,*

  $$[\alpha, f\beta] = f[\alpha,\beta] + \mathcal{L}_{\rho(\alpha)}(f)\beta.$$

In other words, the setions of the bundle act on smooth funtions by derivations via the anchor in suh a way that brakets act as ommutators, and the behavior of the braket with respet to multipliation by funtions is governed by the Leibniz rule. Thus, Lie algebroids are a straightforward generalization of the tangent bundle.

Example 2. *(Tangent bundles). One of the extreme examples of a Lie algebroid over M is the tangent bundle* $A = TM$*, with the identity map as anchor, and the usual Lie bracket of vector fields. Here the isotropy Lie algebras are trivial and the Lie algebroid is transitive.*

Example 3. *(Foliations). Any integrable sub-bundle of the tangent bundle defines a Lie algebroid, choosing the anchor map.*

Example 4. *(The Atiyah sequence) Let $\pi : P \to M$ be a principal G-bundle on the manifold M. Then G-invariant vector fields on P are given by sections of the vector bundle $TP/G \to M$. This bundle has a Lie algebroid structure defined by the Lie bracket on $C^\infty(TP)$ and the surjective anchor π_*, which defines an exact sequence of vector bundles on M:*

$$0 \longrightarrow \mathfrak{g} \longrightarrow TP/G \xrightarrow{\pi_*} T \longrightarrow 0$$

where $\mathfrak{g}$ is the adjoint bundle associated to P.

4 Generalized geometry and Courant algebroids

First let us describe the geometry of $TM \oplus T^*M$. Given a manifold M of dimension m, the direct sum $TM \oplus T^*M$ is equipped with the following canonical structures, let smooth sections $X + \xi, Y + \eta \in TM \oplus T^*M$

- $\langle X + \xi, Y + \eta \rangle := \eta(X) + \xi(Y)$ **Fiberwise inner product**
- $[\![X + \xi, Y + \eta]\!] := [X, Y] + \mathcal{L}_X \eta - i_Y d\xi$ **Courant bracket**.

The vector bundle $TM \oplus T^*M$ with these operations, should be thought of as a **generalized tangent bundle**. A **Dirac structure** on M is a subbundle $L \subseteq TM \oplus T^*M$ satisfying

i) L maximal isotropic with respect to fiberwise inner product, i.e. ($L = L^\perp$, and it has maximal rank)

ii) $[\![\Gamma(L), \Gamma(L)]\!] \subset \Gamma(L)$ i.e. Courant involutive.

Now let us introduce some examples of Dirac structures,

Example 5. *Let $\omega \in \Omega^2(M)$ be a closed 2-form on M, with*

$$\omega^\# : TM \to T^*M, \qquad X \to \omega(X, \cdot).$$

The $Graph(\omega^\#) := \{(X, \omega^\#(X));\quad X \in TM\} \subset TM \oplus T^*M$ is a Dirac structure, indeed, let $X, Y \in TM$

- *$L_\omega = Graph(\omega^\#)$ is maximal isotropic,*

$$\begin{aligned} \langle X + \omega^\#(X), Y + \omega^\#(Y) \rangle &= \omega^\#(Y)(X) + \omega^\#(X)(Y) \\ &= \omega(Y, X) + \omega(X, Y) \\ &= \omega(Y, X) - \omega(Y, X) = 0 \end{aligned}$$

  as L_ω is is a subbundle of $TM \oplus T^*M$, rank(L_ω)= $\dim M$.

- *L_ω is Courant involutive,*

$$\begin{aligned} [\![X + \omega^\#(X), Y + \omega^\#(Y)]\!] &= [X, Y] + \mathcal{L}_X \omega^\#(Y) - i_Y d\omega^\#(X) \\ &= [X, Y] + \mathcal{L}_X i_Y \omega - i_Y d i_X \omega - i_Y i_X d\omega \\ &= [X, Y] + \mathcal{L}_X i_Y \omega - i_Y \mathcal{L}_X \omega \\ &= [X, Y] + [\mathcal{L}_X, i_Y]\omega \\ &= [X, Y] + i_{[X,Y]}\omega = [X, Y] + \omega^\#([X, Y]) \in \Gamma(L_\omega), \end{aligned}$$

  this condition is amount to $d\omega = 0$.

Example 6. *Let $\pi \in \mathfrak{X}^2(M)$ be a Poisson bivector field, with*

$$\pi^\# : T^*M \to TM, \qquad \alpha \to \pi(\alpha, \cdot)$$

We claim that $L_\pi = Graph(\pi^\#) := \{(\pi^\#(\alpha), \alpha);\quad \alpha \in T^*M\} \subset TM \oplus T^*M$ is a Dirac structure, indeed, let $\alpha, \beta \in T^*M$

- L_π *is maximal isotropic,*

$$\begin{aligned}\langle \pi^{\#}(\alpha)+\alpha, \pi^{\#}(\beta)+\beta\rangle &= \beta(\pi^{\#}(\alpha))+\alpha(\pi^{\#}(\beta))\\ &= \pi(\alpha,\beta)+\pi(\beta,\alpha)\\ &= 0\end{aligned}$$

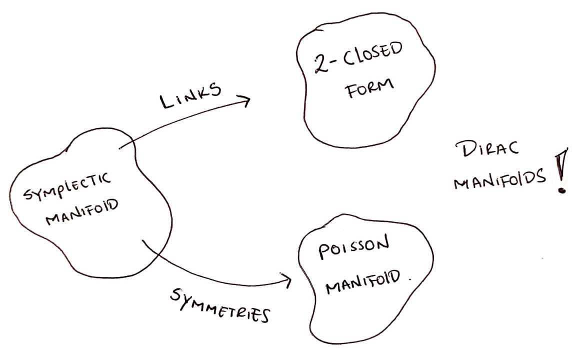
and beyond that Im $\pi^{\#} \subset TM$.

- L_π *is Courant involutive,*

$$\begin{aligned}[\![\pi^{\#}(\alpha)+\alpha, \pi^{\#}(\beta)+\beta]\!] &= [\pi^{\#}(\alpha), \pi^{\#}(\beta)] + \mathcal{L}_{\pi^{\#}(\alpha)}\beta - i_{\pi^{\#}(\beta)}d\alpha\\ &= \pi^{\#}([\alpha,\beta]_\pi) + \mathcal{L}_{\pi^{\#}(\alpha)}\beta - \mathcal{L}_{\pi^{\#}(\beta)}\alpha + di_{\pi^{\#}(\beta)}\alpha\\ &= \pi^{\#}([\alpha,\beta]_\pi) + \mathcal{L}_{\pi^{\#}(\alpha)}\beta - \mathcal{L}_{\pi^{\#}(\beta)}\alpha + d\pi(\alpha,\beta)\\ &= \pi^{\#}([\alpha,\beta]_\pi) + [\alpha,\beta]_\pi \in \Gamma(L_\pi).\end{aligned}$$

this condition is amount to π *being a poisson bivector field.*

Remark 1. *In classical mechanics we have two important constructions. Firstly, suppose we have a symplectic form, and* N *a submanifold of* M *, note that* $i^*\omega \in \Omega^2(N)$ *is closed but it is not necessarily nondegenerate, that is it can stop being a symplectic manifold. This is called in classical mechanical as "links". Secondly, given a symplectic action* $G \curvearrowright (M,\omega)$*, free and proper, note that ,* $\frac{M}{G}$ *it is not necessarily a symplectic manifold, but a Poisson manifold , "symmetries", note that these two constructions links and symmetries, are Dirac manifolds, that was the motivation of Ted Courant.*



All these definitions clearly carry over to the complexified bundle $(TM \oplus T^*M) \otimes \mathbb{C}$, leading to complex Dirac structures. Consider a special class of complex Dirac structures, a **generalized complex structure** is a property of a differential manifold that includes as special cases a complex structure and a symplectic structure. Generalized complex structures were introduced by Nigel Hitchin in 2002 and further developed by his students Marco Gualtieri and Gil Cavalcanti (see [7]). A complex structure on TM is an endomorphism $J : TM \to TM$ satisfying $J^2 = -1$, and, a symplectic structure on TM can be defined as an isomorphism $\omega : TM \to T^*M$ satisfying $\omega^* = -\omega$, with ω a nondegenerate skew form $\omega \in \Lambda^2 TM$. We are using an asterisk to denote the linear dual of a space or mapping, so that ω^* maps $(TM^*)^* = TM$ to TM^*.

In attempting to include both these structures in a higher algebraic structure, we will consider endomorphisms of the direct sum $TM \oplus T^*M$.

Definition 4. *A* ***generalized almost complex structure*** *on* TM *is an endomorphism* $\mathcal{J}$ *of the direct sum* $TM \oplus T^*M$ *which satisfies two conditions. First, it is complex, i.e.* $\mathcal{J}^2 = -1$*; and second, it is symplectic, i.e.* $\mathcal{J}^* = -\mathcal{J}$.

Equivalently, we could define a generalized almost complex structure on TM as an almost complex structure on $TM \oplus T^*M$ which is orthogonal in the natural inner product, i.e. $\mathcal{J}$ satisfies $\mathcal{J}^2 = -1$ and $\mathcal{J}^*\mathcal{J} = 1$.

Example 7. *A) Consider the endomorphism*

$$\mathcal{J}_J = \begin{pmatrix} -J & 0 \\ 0 & J^* \end{pmatrix}, \tag{4.1}$$

where $J : TM \to TM$ is a usual almost complex structure on TM, and the matrix is written with respect to the direct sum $TM \oplus T^*M$. Note that,

$$\mathcal{J}_J^2 = \begin{pmatrix} -J & 0 \\ 0 & J^* \end{pmatrix} \begin{pmatrix} -J & 0 \\ 0 & J^* \end{pmatrix} = \begin{pmatrix} J^2 & 0 \\ 0 & J^{*2} \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

and,

$$\mathcal{J}_J^* = \begin{pmatrix} J & 0 \\ 0 & -J^* \end{pmatrix} = -\begin{pmatrix} -J & 0 \\ 0 & J^* \end{pmatrix} = -\mathcal{J}_J$$

therefore, $\mathcal{J}_J$ is a generalized almost complex structure.

B) Similarly, consider the endomorphism

$$\mathcal{J}_\omega = \begin{pmatrix} 0 & -\omega^{-1} \\ \omega & 0 \end{pmatrix} \tag{4.2}$$

Where $\omega : TM \to T^*M$ is a usual nondegenerate 2-form, $\omega^* = -\omega$. Note that,

$$\mathcal{J}_\omega^2 = \begin{pmatrix} 0 & -\omega^{-1} \\ \omega & 0 \end{pmatrix} \begin{pmatrix} 0 & -\omega^{-1} \\ \omega & 0 \end{pmatrix} = \begin{pmatrix} -\omega^{-1}\omega & 0 \\ 0 & -\omega\omega^{-1} \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

and,

$$\mathcal{J}_\omega^* = \begin{pmatrix} 0 & -\omega^{-1*} \\ \omega^* & 0 \end{pmatrix} = \begin{pmatrix} 0 & \omega^{-1} \\ -\omega & 0 \end{pmatrix} = -\mathcal{J}_\omega$$

therefore, $\mathcal{J}_\omega$ is a generalized almost complex structure.

Remark 2. *A generalized almost complex structure on M is equivalent to the specification of a maximal isotropic complex subbudle $L < (TM \oplus T^*M) \otimes \mathbb{C}$ of real index zero, i.e. such that $L \cap \overline{L} = \{0\}$. This means that studying generalized complex structures is equivalent to studying complex maximal isotropics with real index zero, which is the most generic possible real index.*

4.1 Exact Courant algebroids

In his study of Dirac structures, a notion which includes both Poisson structures and closed 2-forms, T. Courant introduced a bracket on the direct sum of vector fields and 1-forms (**Courant bracket**). This bracket does not satisfy the Jacobi identity except on certain subspaces. In this work we systematize the properties of this bracket in the definition of a Courant algebroid.

A **Courant algebroid** $(E \to M, \langle\cdot,\cdot\rangle, [\![\cdot,\cdot]\!])$ over a manifold M is a vector bundle $E \to M$ equipped with a fibrewise nondegenerate symmetric bilinear form $\langle\cdot,\cdot\rangle$, a bilinear bracket $[\![\cdot,\cdot]\!]$ on the smooth sections $\Gamma(E)$, and a bundle map $\pi : E \to TM$ (called the anchor), such that, for all $e_1, e_2, e_3 \in \Gamma(E)$ and $f \in C^\infty(M)$, the following properties are satisfied:

- C1) $[\![e_1, [\![e_2, e_3]\!]]\!] = [\![[\![e_1, e_2]\!], e_3]\!] + [\![e_2, [\![e_1, e_3]\!]]\!]$
- C2) $[\![e_1, fe_2]\!] = f[\![e_1, e_2]\!] + (\mathcal{L}_{\pi(e_1)}f)e_2$
- C3) $\mathcal{L}_{\pi(e_1)}\langle e_2, e_3\rangle = \langle [\![e_1, e_2]\!], e_3\rangle + \langle e_2, [\![e_1, e_3]\!]\rangle$
- C4) $\pi([\![e_1, e_2]\!]) = [\pi(e_1), \pi(e_2)]$
- C5) $[\![e_1, e_1]\!] = \frac{1}{2}\pi^* d\langle e_1, e_1\rangle$

where in $(C5)$ we identify $E \cong E^*$ via $\langle \cdot, \cdot \rangle$ in order to view π^* as taking values in E.

Definition 5. *Let E be a Courant algebroid. A subbundle L of E is called* ***isotropic*** *if it is isotropic under the symmetric bilinear form $\langle \cdot, \cdot \rangle$ It is called* ***integrable*** *if $\Gamma(L)$ is closed under the bracket $[\![\cdot, \cdot]\!]$ A* ***Dirac structure****, or* ***Dirac subbundle****, is a subbundle L which is maximally isotropic and integrable.*

Example 8. *The model example of a Courant algebroid is $TM \oplus T^*M$ with pairing $\langle X+\xi, Y+\eta \rangle := \eta(X)+\xi(Y)$ where $X+\xi, Y+\eta \in TM \oplus T^*M$, anchor given by the canonical projection $TM \oplus T^*M \to TM$ and bracket given by the H-twisted Courant bracket*

$$[\![X+\xi, Y+\eta]\!]_H = [X,Y] + \mathcal{L}_X \eta - i_Y d\xi + i_Y i_X H \tag{4.3}$$

with H a closed 3-form on M (geometrical structures which are "twisted" by $H \in \Omega^3_{cl}(M)$).

Remark 3. *With this we can extend the concepts of Dirac structures, generalized complex structures, etc. to Courant's general algebroids.*

Now we can introduce what is an exact Courant algebroid. Note that the property $(C5)$ prevents the bracket $[\![\cdot, \cdot]\!]$ from being skew-symmetric, and it implies that $\pi \circ \pi^* = 0$, so we have a chain complex

$$0 \longrightarrow T^*M \xrightarrow{\pi^*} E \xrightarrow{\pi} TM \longrightarrow 0. \tag{4.4}$$

An **exact Courant alegebroids** is a courant algebroid for which the previous sequence is exact, in this case, we always identify T^*M with a subspace of E via π^*. If we have an exact Courant algebroid $E \to M$, we can always choose a right splitting

$$\nabla : TM \to E$$

of the chain complex (4.4) which is isotropic, i.e., whose image in E is isotropic with respect to $\langle \cdot, \cdot \rangle$. Each such ∇ defines a curvature 3-form $H \in \Omega^3_{cl}(M)$ by

$$H(X,Y,Z) := \langle [\![\nabla(X), \nabla(Y)]\!], \nabla(Z) \rangle \tag{4.5}$$

for $X,Y,Z \in \Gamma(TM)$. By using the vector bundle isomorphism $\nabla + \pi^* : TM \oplus T^*M \to E$, the Courant algebroid structure on E can be identified with the usual Courant algebroid structure on $TM \oplus T^*M$ defined by (4.3)

Example 9. *Let $\mathfrak{g}$ be a Lie algebra with a non-degenerate invariant symmetric pairing $\langle \cdot, \cdot \rangle$ (i.e. $\mathfrak{g}$ is a Courant algebroid over a point), G a connected Lie group integrating $\mathfrak{g}$, an $H \subset G$ a Lie subgroup such that $\mathfrak{h}^\perp = \mathfrak{h}$ (i.e. $\mathfrak{h} \subset \mathfrak{g}$ is a Lagrangian Lie subalgebra). Then the trivial vector bundle $\mathfrak{g} \times G/H \to G/H$ is naturally an exact Courant algebroid: the pairing and the bracket of constant sections are the same as in $\mathfrak{g}$, and the anchor map π is the action of $\mathfrak{g}$ on G/H. Exact Courant algebroids of this type play an important role in Poisson-Lie T-duality.*

Remark 4. *Exact Courant algebroids are classified by $H^3(M,R)$: if we split the exact sequence (4.4) so that $TM \subset TM \oplus T^*M \cong E$ is $\langle \cdot, \cdot \rangle$-isotropic then the 3-form $H \in \Omega^3(M)$ given by (4.5) in addition to being closed, its cohomology class is independent of the splitting and completely determines the exact Courant algebroid E up to isomorphism. This classification of Courant algebroids were first studied by Pavol Ševera, we call $[H]$ the Ševera class of E. When this class is integral, the exact Courant algebroid may be viewed as a generalized Atiyah sequence associated to a connection on an S^1 gerbe. In this sense, exact Courant algebroids arise naturally from the study of gerbes.*

5 Extended actions on courant algebroids

Let a Lie group G act on a manifold M, so that we have the Lie algebra homomorphism $\mathfrak{g} \to \Gamma(TM)$. We wish to extend this action to a Courant algebroid E, making E into a G-equivariant vector bundle, in such a way that the Courant algebroid structure is preserved. Recall that an infinitesimal automorphism of a vector bundle E is a pair (F, X), where $X \in \Gamma(TM)$ and $F : \Gamma(E) \to \Gamma(E)$ satisfies

$$F(fe) = fF(e) + (\mathcal{L}_X f)e, \quad e \in \Gamma(E), f \in C^\infty(M). \tag{5.1}$$

Infinitesimal bundle automorphisms form a Lie algebra with respect to the bracket

$$[(F_1, X_1), (F_2, X_2)] := (F_1F_2 - F_2F_1, [X_1, X_2]),$$

in fact, this is of course bilinear. Let us check that it satisfies jacobi identity :

$$\begin{aligned}
&[[(F_1, X_1), (F_2, X_2)], (F_3, X_3)] + [[(F_3, X_3), (F_1, X_1)], (F_2, X_2)] + [[(F_2, X_2), (F_3, X_3)], (F_1, X_1)] \\
=&[(F_1F_2 - F_2F_1, [X_1, X_2]), (F_3, X_3)] + [(F_3F_1 - F_1F_3, [X_3, X_1]), (F_2, X_2)] + [(F_2F_3 - F_3F_2, [X_2, X_3]), (F_1, X_1)] \\
=&((F_1F_2 - F_2F_1)F_3 - F_3(F_1F_2 - F_2F_1), [[X_1, X_2], X_3]) \\
&+ ((F_3F_1 - F_1F_3)F_2 - F_2(F_3F_1 - F_1F_3), [[X_3, X_1], X_2]) \\
&+ ((F_2F_3 - F_3F_2)F_1 - F_1(F_2F_3 - F_3F_2), [[X_2, X_3], X_1]) \\
=&(0, [[X_1, X_2], X_3] + [[X_3, X_1], X_2] + [[X_2, X_3], X_1])=(0, 0)
\end{aligned}$$

and $[(F, X), (F, X)] = (FF - FF, [X, X]) = (0, 0)$.

If E is a Courant algebroid, then its Lie algebra of symmetries, denoted by $\mathbf{sym}(E)$, consists of infinitesimal bundle automorphism (F, X) which preserve the bracket $[\![\cdot, \cdot]\!]$, the pairing $\langle \cdot, \cdot \rangle$ and the anchor $\pi : E \to TM$:

$$F([\![e_1, e_2]\!]) = [\![F(e_1), e_2]\!] + [\![e_1, F(e_2)]\!]$$

$$\mathcal{L}_X \langle e_1, e_2 \rangle = \langle F(e_1), e_2 \rangle + \langle e_1, F(e_2) \rangle$$

$$\pi \circ F = \mathcal{L}_X \circ \pi$$

where $e_1, e_2 \in \Gamma(E)$. Let $e \in \Gamma(E)$, the infinitesimal bundle automorphism $([[\![e, \cdot]\!], \pi(e)])$ belongs to $\mathbf{sym}(E)$. In fact, applying the properties $C1) - C4)$, we have

$$\begin{aligned}
F([\![e_1, e_2]\!]) = [\![e, [\![e_1, e_2]\!]]\!] &= [\![[\![e, e_1]\!], e_2]\!] + [\![e_1, [\![e, e_2]\!]]\!] \\
&= [\![F(e_1), e_2]\!] + [\![e_1, F(e_2)]\!]
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{L}_X \langle e_1, e_2 \rangle = \mathcal{L}_{\pi(e)} \langle e_1, e_2 \rangle &= \langle [\![e, e_1]\!], e_2 \rangle + \langle e_1, [\![e, e_2]\!] \rangle \\
&= \langle F(e_1), e_2 \rangle + \langle e_1, F(e_2) \rangle.
\end{aligned}$$

Moreover

$$\pi \circ F = \pi([\![e, \cdot]\!]) = \mathcal{L}_{\pi(E)} \pi.$$

As a result, we obtain a map:

$$\mathrm{ad} : \Gamma(E) \to \mathbf{sym}(E), \qquad e \to ([\![e, \cdot]\!], \pi(e)). \tag{5.2}$$

The elements in the image of ad are called **inner symmetries** of E. Note that the map ad extends the usual identification of vector fields $X \in \Gamma(TM)$ with infinitesimal symmetries of the Lie bracket on TM:

$$\Gamma(TM) \to \mathbf{sym}(TM), \qquad X \to ([X, \cdot], X). \tag{5.3}$$

Let $\mathfrak{g}$ be a Lie algebra , **an equivariant structure** on E preserving its Courant algebroid structure is defined infinitesimally by an algebra homomorphism Lie $\mathfrak{g} \to \mathbf{sym}(E)$, where we have $\mathfrak{g}$-actions by inner symmetries, i.e. compositions

$$\mathfrak{g} \xrightarrow{\psi} \Gamma(E) \xrightarrow{\mathrm{ad}} \mathbf{sym}(E). \tag{5.4}$$

Since the Courant bracket on $\Gamma(E)$ is not a Lie bracket, it is natural to replace the Lie algebra $\mathfrak{g}$ in (5.4) by a more general structure with "Courant-type" bracket. We call these as Courant algebras and describe them below.

5.1 Courant algebras

In the present subsection, we recall the definitions of a Courant algebra and exact Courant algebra. Let $\mathfrak{g}$ be a Lie algebra.

Definition 6. *A **Courant algebra** over the Lie algebra $\mathfrak{g}$ is a vector space $\mathfrak{a}$ equipped with a bilinear bracket $[\![\cdot,\cdot]\!] : \mathfrak{a}\times\mathfrak{a}\to\mathfrak{a}$ and a map $\pi:\mathfrak{a}\to\mathfrak{g}$ which satisfy the following conditions for all $a_1, a_2, a_3 \in \mathfrak{a}$:*

$$[\![a_1,[\![a_2,a_3]\!]]\!] = [\![[\![a_1,a_2]\!],a_3]\!] + [\![a_2,[\![a_1,a_3]\!]]\!] \tag{5.5}$$

$$\pi([\![a_1,a_2]\!]) = [\pi(a_1),\pi(a_2)]. \tag{5.6}$$

In other words, $\mathfrak{a}$ is a Leibniz algebra with a homomorphism π from $\mathfrak{a}$ to $\mathfrak{g}$. We will say that $\pi:\mathfrak{a}\to\mathfrak{g}$ denotes a Courant algebra.

A Courant algebroid over a smooth manifold M gives an example of a Courant algebra over the Lie algebra of vector fields $\mathfrak{g}=\Gamma(TM)$ by taking $\mathfrak{a}$ as the Leibniz algebra structure on the space of sections of the underlying vector bundle of the Courant algebroid.

Definition 7. *An **exact Courant algebra** over the Lie algebra $\mathfrak{g}$ is a Courant algebra $\pi:\mathfrak{a}\to\mathfrak{g}$ for which π is a surjective linear map and $\mathfrak{h}=ker(\pi)$ is abelian, i.e. $[h_1,h_2]=0$ for all $h_1,h_2\in\mathfrak{h}$.*

For an exact Courant algebra $\pi:\mathfrak{a}\to\mathfrak{g}$, there are actions of $\mathfrak{g}$ on $\mathfrak{h}$: $[g,h]=[a,h]$ and $[h,g]=[h,a]$, for any a such that $\pi(a)=g$. This defines $\mathfrak{h}$ as a representation of $\mathfrak{g}$ viewed as a Leibniz algebra.
The next example will give a natural exact Courant algebra associated with any representation of the Lie algebra $\mathfrak{g}$.

Example 10 (Hemisemidirect product)**.** *If $\mathfrak{g}$ is a Lie algebra and $\mathfrak{h}$ is a $\mathfrak{g}$-module, then we can endow $\mathfrak{a}:=\mathfrak{g}\oplus\mathfrak{h}$ with the structure of an exact Courant algebra by taking $\pi:\mathfrak{g}\oplus\mathfrak{h}\to\mathfrak{g}$ to be the natural projection and defining*

$$[\![(u_1,w_1),(u_2,w_2)]\!] = ([u_1,u_2], u_1\cdot w_2). \tag{5.7}$$

Since π is a projection then is surjective, In addition, that π preserves the brackets, and that $[\![\mathfrak{h},\mathfrak{h}]\!]=0$. Finally, condition (5.5) is a consequence of the Jacobi identity for $\mathfrak{g}$ and the fact that $\mathfrak{h}$ is a $\mathfrak{g}$-module. In fact,

$$\begin{aligned}
&[\![(u_1,w_1),[\![(u_2,w_2),(u_3,w_3)]\!]]\!] - [\![[\![(u_1,w_1),(u_2,w_2)]\!],(u_3,w_3)]\!] - [\![(u_2,w_2),[\![(u_1,w_1),(u_3,w_3)]\!]]\!]\\
&= [\![(u_1,w_1),([u_2,u_3],u_2\cdot w_3)]\!] - [\![([u_1,u_2],u_1\cdot w_2),(u_3,w_3)]\!] - [\![(u_2,w_2),([u_1,u_3],u_1\cdot w_3)]\!]\\
&= ([u_1,[u_2,u_3]],u_1\cdot u_2\cdot w_3) - ([[u_1,u_2],u_3],[u_1,u_2]\cdot w_3) - ([u_2,[u_1,u_3]],u_2\cdot u_1\cdot w_3)\\
&\overset{\bigstar}{=} ([u_1,[u_2,u_3]] - [[u_1,u_2],u_3] - [u_2,[u_1,u_3]], u_1\cdot u_2\cdot w_3 - [u_1,u_2]\cdot w_3 - u_2\cdot u_1\cdot w_3)\\
&= (0,0)
\end{aligned} \tag{5.8}$$

In the first component of $\bigstar$ we used that the Jacobi condition is satisfied for $\mathfrak{g}$, and in the second component the fact that $\mathfrak{h}$ is a $\mathfrak{g}$-module. where $u\cdot w$ denotes the $\mathfrak{g}$-action.

5.2 Extended actions

Let a Lie group G act on a manifold M. We will denote a Courant algebra morphism simply by $\Psi:\mathfrak{g}\to\Gamma(TM)$, keeping in mind that it always projects to an action on M, denoted by $\psi:\mathfrak{g}\to\Gamma(TM)$, we wish to extend this action to a Courant algebroid E, making E into a G-equivariant vector bundle, in such a way that the Courant algebroid structure is preserved.

Definition 8. *Let G be a connected Lie group acting on a manifold M with infinitesimal action $\psi:\mathfrak{g}\to\Gamma(TM)$. An **extended $\mathfrak{g}$-action** on an exact Courant algebroid E over M is a Courant algebra morphism $\Psi:\mathfrak{a}\to\Gamma(E)$ from an exact Courant algebra $\mathfrak{h}\to\mathfrak{a}\to\mathfrak{g}$ into $\Gamma(E)$*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathfrak{h} & \longrightarrow & a & \longrightarrow & \mathfrak{g} & \longrightarrow & 0 \\ & & & & \Psi \big\downarrow & & \big\downarrow \psi & & \\ & & & & \Gamma(E) & \longrightarrow & \Gamma(TM) & & \end{array}$$

so that $\Psi(\mathfrak{h}) \subseteq \Omega^1(M)$. We call it an ***extended G-action*** *if the induced G-action on E integrates to a G-action. In particular, an extended G-action on E makes it into an equivariant G-bundle, with G acting by Courant algebroid automorphisms.*

Remark 5. *Let $\Psi : \mathfrak{a} \to \Gamma(E)$ be an extended $\mathfrak{g}$-action for which the projected action $\psi : \mathfrak{g} \to \Gamma(TM)$ integrates to a global G-action on M. In this case, a sufficient condition ensuring that this data defines an extended G-action on E (and not only an extended action of a cover of G) is the existence of a $\mathfrak{g}$-invariant isotropic splitting for E (these always exist, e.g., if G is compact). Indeed, such a split exact Courant algebroid is isomorphic to the direct sum $TM \oplus T^*M$, equipped with the H-twisted Courant bracket for a closed 3-form H. In this splitting, therefore, for each $a \in \mathfrak{a}$ the section $\Psi(a)$ decomposes as $\Psi(a) = X_a + \xi_a$, and it acts via $[\![X_a + \xi_a, Y + \eta]\!]_H = [X_a, Y] + \mathcal{L}_{X_a}\eta - i_Y d\xi_a + i_Y i_{X_a} H$, or as a matrix*

$$ad_{\Psi(a)} = \begin{pmatrix} \mathcal{L}_{X_a} & 0 \\ i_{X_a}H - d\xi_a & \mathcal{L}_{X_a} \end{pmatrix}$$

We see immediately from this that the splitting is preserved by this action if and only if for each $a \in \mathfrak{a}$,

$$i_{X_a}H = d\xi_a \tag{5.9}$$

Indeed, such a splitting gives an identification of E with the Courant algebroid $(TM \oplus T^*M, \langle\cdot,\cdot\rangle, [\![\cdot,\cdot]\!]_H)$. Since M is a G-manifold, $TM \oplus T^*M$ is naturally a G-equivariant bundle, and condition (5.9) exactly says that this canonical G-action on $TM \oplus T^*M$ coincides infinitesimally with the one induced by under the identification $E \cong TM \oplus T^*M$.

We now provide a complete description, assuming G to be compact and E exact, of the simplest kind of extended action, namely one for which $\mathfrak{a} = \mathfrak{g}$.

Definition 9. *A* ***trivially extended G-action*** *is one for which $\mathfrak{a} = \mathfrak{g}$ and $\pi : \mathfrak{a} \to \mathfrak{g}$ is the identity map, as described by the commutative diagram*

$$\begin{array}{ccc} \mathfrak{g} & \xrightarrow{\text{Id}} & \mathfrak{g} \\ \big\downarrow \widetilde{\psi} & & \big\downarrow \psi \\ \Gamma(E) & \xrightarrow{\pi} & \Gamma(TM), \end{array}$$

This is also called a lifted action on E, and denoted by $\widetilde{\psi} : \mathfrak{g} \to \Gamma(E)$.

A trivial example is of course when $E = TM \oplus T^*M$, with $H = 0$, and e $\widetilde{\psi} = \psi$ is an ordinary action.

Example 11 (Lifted actions)**.** *Take E to be the Courant algebroid $TM \oplus T^*M$, with $H = 0$, let $\psi : \mathfrak{g} \to \Gamma(TM)$ be an action on M, and $v : M \to \mathfrak{g}^*$ be an equivariant map. Then*

$$\widetilde{\psi}(u) := \psi(u) + d\langle v, u\rangle, \;\; u \in \mathfrak{g}$$

is a lifted action on E.

5.3 Moment maps for extended actions

Suppose that we have an extended $\mathfrak{g}$-action $\Psi : \mathfrak{a} \to \Gamma(E)$ on an exact Courant algebroid E over M, and $\mathfrak{h} \to \mathfrak{a} \to \mathfrak{g}$ be an exact Courant algebra. Recall that this implies that $\Psi(\mathfrak{a}) \subset \Omega^1_{cl}(M)$, and that $\mathfrak{h}$ is a $\mathfrak{g}$-module. Let us equip $\mathfrak{h}^*$ with the dual $\mathfrak{g}$-action.

Definition 10. *A moment map for an extended $\mathfrak{g}$-action on an exact Courant algebroid is a $\mathfrak{g}$-equivariant map $\mu : \mathfrak{h} \to C^\infty(M)$ satisfying $d\mu = v$, i.e. such that the following diagram commutes:*

$$\begin{array}{ccc} & & \mathfrak{h} \\ & \overset{\mu}{\swarrow} & \downarrow \nu \\ C^\infty(M) & \xrightarrow{\ d\ } & \Gamma(T^*M) \end{array}$$

Note that μ may be alternatively viewed as an equivariant map $\mu : M \to \mathfrak{h}^*$ such that, for each $w \in \mathfrak{h}$, $\Psi(w) = d\langle \mu, w\rangle$.

Proposition 1. *Let $\widetilde{\psi} : \mathfrak{g} \to \Gamma(E)$ be an be an isotropic lifted $\mathfrak{g}$-action on an exact Courant algebroid E, $\mathfrak{h}$ be a $\mathfrak{g}$-module, and $\mu : M \to \mathfrak{h}^*$ be an equivariant map. Then the map $\Psi : \mathfrak{g} \oplus \mathfrak{h} \to \Gamma(E)$,*

$$\Psi(u, w) = \widetilde{\psi}(u) + d\langle \mu, w\rangle, \tag{5.10}$$

defines an extended $\mathfrak{g}$-action of the hemisemidirect product $\mathfrak{a} = \mathfrak{g} \oplus \mathfrak{h}$ on E with moment map μ. Moreover, the image $\Psi(\mathfrak{a}) \subseteq E$ is isotropic over $\mu^{-1}(0)$.

Proof: Let Ψ be defined as in (5.10). Then, using that $[\![\xi, \cdot]\!] = 0$ if $\xi \in \Omega^1_{cl}(M)$, we get

$$\begin{aligned}
[\![\Psi(u_1, w_1), \Psi(u_2, w_2)]\!] &= [\![\widetilde{\psi}(u_1) + d\langle \mu, w_1\rangle, \widetilde{\psi}(u_2) + d\langle \mu, w_2\rangle]\!] \\
&= [\![\widetilde{\psi}(u_1), \widetilde{\psi}(u_2)]\!] + [\![\widetilde{\psi}(u_1), d\langle \mu, w_2\rangle]\!] + [\![d\langle \mu, w_1\rangle, \widetilde{\psi}(u_2)]\!] + [\![d\langle \mu, w_1\rangle, d\langle \mu, w_2\rangle]\!] \\
&= [\![\widetilde{\psi}(u_1), \widetilde{\psi}(u_2)]\!] + [\![\widetilde{\psi}(u_1), d\langle \mu, w_2\rangle]\!] \\
&= \widetilde{\psi}([u_1, u_2]) + \mathcal{L}_{\psi(u_1)} d\langle \mu, w_2\rangle \\
&= \widetilde{\psi}([u_1, u_2]) + d\mathcal{L}_{\psi(u_1)} \langle \mu, w_2\rangle \\
&= \widetilde{\psi}([u_1, u_2]) + d\langle \mu, u_1 \cdot w_2\rangle \\
&= \Psi([u_1, u_2], u_1 \cdot w_2)
\end{aligned}$$

where for the last equality we used the equivariance of μ. Comparing with (5.7), we conclude that preserves brackets. So it defines a Courant algebra morphism. It is also clear that $\Psi(\mathfrak{h}) \subseteq \Omega^1_{cl}(M)$, hence Ψ is an extended $\mathfrak{g}$-action. Let us now consider the pairing

$$\begin{aligned}
\langle \Psi(u_1, w_1) =, \Psi(u_2, w_2)\rangle &= \langle \widetilde{\psi}(u_1), d\langle \mu, w_2\rangle\rangle + \langle \widetilde{\psi}(u_2), d\langle \mu, w_1\rangle\rangle \\
&= \mathcal{L}_{\psi(u_1)} \langle \mu, w_2\rangle + \mathcal{L}_{\psi(u_2)} \langle \mu, w_1\rangle \\
&= \langle \mu, u_1 \cdot w_2\rangle + \langle \mu, u_2 \cdot w_1\rangle
\end{aligned}$$

which vanishes on points $x \in M$ where $\mu(x) = 0$.

Example 12. *Let G be a Lie group acting on a symplectic manifold (M, ω) preserving the symplectic form, and let $\psi : \mathfrak{g} \to \Gamma(TM)$ denote the infinitesimal action. We now show that there is a natural extended action of the hemisemidirect product Courant algebra $\mathfrak{g} \oplus \mathfrak{g}$ on the standard Courant algebroid $TM \oplus T^*M$ with $H = 0$. As described in Example (10), the Courant algebra is described by the sequence*

$$0 \longrightarrow \mathfrak{g} \longrightarrow \mathfrak{g} \oplus \mathfrak{g} \xrightarrow{\ \pi\ } \mathfrak{g} \longrightarrow 0,$$

and is equipped with the bracket

$$[(g_1, h_1), (g_2, h_2)] = ([g_1, g_2], [g_1, h_2]).$$

Then define the action $\rho : \mathfrak{g} \oplus \mathfrak{g} \to \Gamma(TM \oplus T^*M)$ by

$$\rho(g, h) = X_g + i_{X_h}\omega,$$

where $X_g = \psi(g)$, for $g \in \mathfrak{g}$, and ω is the symplectic form. Thus, since

$$[X_{g_1} + i_{X_{h_1}}\omega, X_{g_2} + i_{X_{h_2}}\omega] = [X_{g_1}, X_{g_2}] + \mathcal{L}_{X_{g_1}} i_{X_{h_2}}\omega = X_{[g_1,g_2]} + i_{X_{[g_1,h_2]}}\omega,$$

we see that ρ is a Courant morphism.

The question of finding a moment map for this extended action then becomes the one of finding an equivariant map $\mu : \mathfrak{g} \to C^\infty(M)$ such that $d(\mu_g) = i_{X_g}\omega$. Hence we recover the usual moment map for a Hamiltonian action on a symplectic manifold.

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